The Big - M Method

The Big - M method is another method of removing artificial variables from the basis. In this method we assign coefficients to artificial variables, undesirable from the objective function. If objective function Z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The penalty will designated by
$-M$ for a maximization problem and $+M$ for a minimization problem, where
$M > 0$.

The following are steps of the Algorithm for solving LPP by the Big - M method;

(i) Express the LPP in the standard form by adding slack variables, surplus variables and artificial variables. Assign a zero coefficient to both slack and surplus variables and a very large positive coefficient $+M$ (for min. case) and $-M$ (for max. case) to artificial variable in the objective function.

(ii) The initial basic feasible solution is obtained by assigning zero value to original variables.

(iii) Calculate the value of $C_j - Z_j$ in last row of simplex table and examine these values.

- If all $C_j - Z_j \geq 0$ then the current basic feasible solution is optimal.
- If for a column k, $C_k - Z_k$ is most negative and all entries in this column are negative, then the problem has unbounded optimal solution.
- If one or more $C_j - Z_j < 0$ (minimization case), then select the variable to enter into the basis with the largest negative $C_j - Z_j$ value. That is $C_k - Z_k = Min\{C_j - Z_j\} : C_j - Z_j < 0$.

(iv) Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Remarks

At any iteration of the simplex algorithm any one of the following cases may arise;

1. If at least one artificial variable is present in the basis with zero coefficient of M in each case $C_j - Z_j \geq 0$, then the given LPP has no solution. That is, the current basic feasible solution is degenerate.
2. If at least one artificial variable is present in the basis with positive value and the coefficient of M in each $C_j - Z_j \geq 0$, then given LPP has no optimum basic feasible solution. In this case the given LPP has a pseudo optimum basic feasible solution.

Example 1: Solve the following LPP using penalty (Big - M) method;

$$Max\ Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and

$$x_1, x_2, x_3 \geq 0$$

Solution

Since all constraints of the given LPP are equation, therefore adding only artificial variables A_1 and A_2 in the constraints. The standard form of the problem becomes;

$$Max\ Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$$

subject to

$$x_1 + 2x_2 + 3x_3 + A_1 = 15$$
$$2x_1 + x_2 + 5x_3 + A_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and

$$x_1, x_2, x_3, A_1, A_2 \geq 0$$

The initial basic feasible solution is given in Table1 below;

Table 1: Initial Solution

		$C_j \longrightarrow$	1	2	3	-1	-M	-M	
C_B	B	$b(=x_B)$	x_1	x_2	x_3	x_4	A_1	A_2	Min.Ratio
-M	A_1	15	1	2	3	0	1	0	$\frac{15}{3} = 5$
-M	A_2	20	2	1	5	0	0	1	$\frac{20}{5} = 4 \rightarrow$
-1	x_4	10	1	2	1	1	0	0	$\frac{10}{1} = 10$
$Z = -35M - 10$		$Z_j =$	-3M-1	-3M-2	-8M-1	-1	-M	-M	
		$C_j - Z_j$	3M+2	3M+4	8M+4 $\uparrow$	0	0	0	

Since the value of $C_3 - Z_3$ in Table 1 has largest positive value the variable x_3 is chosen to enter into the basis. To get an improved basic feasible solution, apply the following row operations and removing A_2 from the basis.

$$R_2(new) \rightarrow \frac{R_2(old)}{5(key\ element)} = (4,\ 2/5,\ 1/5,\ 1,\ 0,\ 0)$$

$$R_1(new) \rightarrow R_1(old) - (3)R_2(new) = (3, -1/5, 7/5, 0, 0, 1)$$

$$R_3(new) \rightarrow R_3(old) - (1)R_1(new) = (6, 3/5, 9/5, 0, 1, 0)$$

The improved solution is shown in Table 2

Table 2: Improved Solution

		$C_j \longrightarrow$	1	2	3	-1	-M	
C_B	B	$b(=x_B)$	x_1	x_2	x_3	x_4	A_1	Min.Ratio
-M	A_1	3	-1/5	7/5	0	0	1	$\frac{3}{7/5} = 15/7 \rightarrow$
3	x_3	4	2/5	1/5	1	0	0	$\frac{4}{1/5} = 20$
-1	x_4	6	3/5	9/5	0	1	0	$\frac{6}{9/5} = 30/9$
$Z = -3M + 6$		$Z_j =$	(M/5)-3/5	-(7M/5)-6/5	3	-1	-M	
		$C_j - Z_j$	-(M/5)-2/5	(7M/5)+16/5 $\uparrow$	0	0	0	

The solution shown in Table 2 is not optimal because $C_2 - Z_2$ is positive. Thus, applying the following row operations for entering variable x_2 into the basis and removing variable A_1 from the basis.

$$R_1(new) \rightarrow \frac{R_1(old)}{7/5(key\,element)} = (15/7, -1/7, 1, 0, 0)$$

$$R_2(new) \rightarrow R_2(old) - (1/5)R_1(new) = (25/7, 3/7, 0, 1, 0)$$

$$R_3(new) \rightarrow R_3(old) - (9/5)R_1(new) = (15/7, 6/5, 0, 0, 1)$$

The new solution is shown in Table 3

Table 3: Improved Solution

		$C_j \longrightarrow$	1	2	3	-1	
C_B	B	$b(=x_B)$	x_1	x_2	x_3	x_4	Min.Ratio
2	x_2	15/7	-1/7	1	0	0	-
3	x_3	25/7	3/7	0	1	0	$\frac{25/7}{3/7} = 25/3$
-1	x_4	15/7	6/7	0	0	1	$\frac{15/7}{6/7} = 5/2 \rightarrow$
$Z = 90/7$		Z_j	1/7	2	3	-1	
		$C_j - Z_j$	6/7 $\uparrow$	0	0	0	

Again, the solution shown in Table 3 is not optimal. Thus, applying the following row operations by entering x_1 into the basis and removing variable x_4 from the basis.

$$R_3(new) \rightarrow \frac{R_3(old)}{6/7(key\ element)} = (15/6, 1, 0, 0, 7/6)$$

$$R_2(new) \rightarrow R_2(old) - (3/7)R_3(new) = (15/6, 0, 0, 1, -1/2)$$
$$R_1(new) \rightarrow R_1(old) - (-1/7)R_3(new) = (15/6, 0, 1, 0, 1/6)$$

The new solution is shown in Table 4

Table 4: Optimal Solution

		$C_j \longrightarrow$	1	2	3	-1
C_B	B	$b(= x_B)$	x_1	x_2	x_3	x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	-1/2
1	x_1	15/6	1	0	0	7/6
$Z =$ 15		Z_j	1	2	3	0
		$C_j - Z_j$	0	0	0	-1

Since all $C_j - Z_j \leq 0$ in Table 4, Thus, an optimal solution has been arrived with values of variables as $x_1 = 15/6, x_2 = 15/6, x_3 = 15/6, x_4 = 0$ and $Max\ Z = 15$.

Example 2: Solve the following LPP using penalty (Big - M) method;

$$Main\ Z = 600x_1 + 500x_2$$

subject to

$$2x_1 + x_2 \geq 80$$
$$x_1 + 2x_2 \geq 60$$

and

$$x_1, x_2 \geq 0$$

Solution:

By introducing surplus variables S_1 and S_2 and artificial variables A_1 and A_2 in the constraints. The standard form of the problem becomes;

$$Main\ \ Z = 600x_1 + 500x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$$

subject to

$$2x_1 + x_2 - S_1 + A_1 = 80$$
$$x_1 + 2x_2 - S_2 + A_2 = 60$$

and

$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

The initial basic feasible solution is obtained by setting $x_1 = x_2 = S_1 = S_2 = 0$ as shown in Table 1;

Table 1: Initial Solution

		$C_j \longrightarrow$	600	500	0	0	M	M	
C_B	B	$b(= x_B)$	x_1	x_2	S_1	S_2	A_1	A_2	Min.Ratio
M	A_1	80	2	1	-1	0	1	0	$\frac{80}{1} = 80$
M	A_2	60	1	2	0	-1	0	1	$\frac{60}{2} = 30 \rightarrow$
$Z = 140M$		Z_j	3M	3M	-M	-M	M	M	
		$C_j - Z_j$	600-3M	500-3M ↑	M	M	0	0	

Since the value of $C_2 - Z_2$ in Table 1 has largest negative value, therefore enter variable x_2 to replace basic variable A_2 into the basis. To get an improved basic feasible solution, apply the following row operations.

$$R_2(new) \rightarrow \frac{R_2(old)}{2(key\,element)} = (30, 1/2, 1, 0, -1/2, 0)$$

$$R_1(new) \to R_1(old) - (1)R_2(new) = (50, 3/2, 0, -1, 1/2, 1)$$

The improved solution is shown in Table 2

Table 2: Improved Solution

		$C_j \longrightarrow$	600	500	0	0	M	
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	A_1	Min.Ratio
M	A_1	50	3/2	0	-1	1/2	1	$\frac{50}{3/2} = 33.33 \to$
500	x_2	30	1/2	1	0	-1/2	0	$\frac{30}{1/2} = 60$
$Z = 15000 + 50M$		Z_j	(3M/2)+250	500	-M	(M/2)-250	M	
		$C_j - Z_j$	350-3M ↑	0	M	250-M/2	0	

The solution shown in Table 2 is not optimal because $C_1 - Z_1$ is largest negative. Thus, applying the following row operations by entering variable x_1 into the basis and removing variable A_1 from the basis.

$$R_1(new) \to \frac{R_1(old)}{3/2(key\ element)} = (100/3, 1, 0, -2/3, 1/3)$$

$$R_2(new) \to R_2(old) - (1/2)R_1(new) = (40/3, 0, 1, 1/3, -2/3)$$

The new solution is shown in Table 3

Table 3 Optimal Solution

		$C_j \longrightarrow$	600	500	0	0
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2
600	x_1	100/3	1	0	-2/3	1/3
500	x_2	40/3	0	1	1/3	-2/3
$Z = 80,000/3$		Z_j	600	500	-700/3	-400/3
		$C_j - Z_j$	0	0	700/3	400/3

In Table 3 all the numbers in the $C_j - Z_j$ row are either zero or positive and also both artificial variables have been reduced to zero, an optimum solution has been arrived at with $x_1 = 100/3, x_2 = 40/3$ and total minimum cost, $Z = 80,000/3$.